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Response variability due to randomness in Poisson's ratio for plane-strain and plane-stress states

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Summary

In the ordinary structural materials, one of the parameters that can be assumed to have spatial uncertainty is Poisson's ratio. Therefore the independent evaluation of the effects of this parameter on the response variability is of importance. The difficulties in obtaining the response variability due to randomness in Poisson's ratio lie in the fact that the Poisson's ratio enters the stiffness matrix as a non-linear parameter. In this paper, a formulation to determine the response variability in plane strain and plane stress states due to the randomness in the Poisson's ratio is given. The formulation is accomplished by means of the stochastic decomposition of the constitutive matrix into several sub-matrices taking into consideration of the polynomial expansion on the coefficients of constitutive relation. To demonstrate the validity of the proposed formulation, some example structures are chosen and the results are compared with those obtained by means of Monte Carlo simulation. Through the formulation proposed in this study, it becomes possible for the weighted integral stochastic finite element analysis to consider all the uncertain material parameters in its application.
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Keywords: Stochastic FEM; Uncertain Poisson's ratio; Weighted integral method; Constitutive relationship; Response variability; Monte Carlo simulation

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1. Introduction

In the majority of research in the stochastic finite element analysis, the stochasticity in elastic modulus is mainly taken into consideration (Bhattacharyya and Chakraborty, 2002; Choi and Noh, 1996; Deodatis and Shinozuka, 1989; Deodatis et al., 1991; Falsone and Impollonia, 2002; Shinozuka and Deodatis, 1988). This is due mainly to the fact that the elastic modulus is a parameter of importance in determining the system behavior and due partly to the fact that the formulation for response variability analysis is relatively simple. However, it is natural to assume that virtually all the parameters in a system have inherent uncertainties in spatial and/or temporal domain (Kleiber and Hein, 1992). As the Poisson's ratio, together with the elastic modulus, is a material constant which influences the behavior of structural systems, the evaluation of the sole effect of this parameter on the response variability is of importance. However, since the Poisson's ratio enters the element stiffness matrix as a non-linear parameter, some difficulties are involved in evaluating the response variability in the non-statistical stochastic finite element analysis (Graham and Deodatis, 2001). In the literature, it is restricted to take into consideration of the randomness in Poisson's ratio indirectly by way of considering the Lamé's parameters (Graham and Deodatis, 2001; Stefanou and Papadrakakis, 2004), which are functions not only of the Poisson's ratio but also of the elastic modulus (Zienkiewicz and Taylor, 1989), as a random material parameter.

The weighted integral stochastic finite element method, one of the first order series expansion methods, is developed by many researchers, and is applied to the in-plane structures having randomness in elastic modulus (Choi and Noh, 1996, 2000; Deodatis et al., 1991; Graham and Deodatis, 2001; Takada, 1990). To improve the statistics in the weighted integral scheme and to overcome the shortcomings of the first order expansion, the effects of higher order terms are taken into account (Choi and Noh, 2000). In the category of perturbation method, new tries had been made to overcome the drawbacks in that method (Elishakoff et al., 1997; Falsone and Impollonia, 2002; Kaminski, 2001). Some research works are dedicated to determine the bounds in response variability (Deodatis and Shinozuka, 1989; Deodatis et al., 2003; Papadopoulos et al., 2005) and to the dynamic and non-linear problems (Adhikari and Manohar, 1999; Anders and Hori, 1999; Galal et al., 2002; Liu et al., 1986; Li et al., 1999). In addition to the material parameters, some researchers put their focus on the evaluation of response variability due to randomness in geometrical parameters such as the thickness of plate structures and section of beams (Altus and Totry, 2003; Choi and Noh, 1996, 2000; Lawrence, 1987), due to temporal uncertainties in applied loads (Chiostrini and Facchini, 1999; Galal et al., 2002; To, 1986) and due to random temperature (Liu et al., 2001) in concrete structures.

In this paper, a formulation to analyze the response variability related to the spatial randomness in Poisson's ratio in the plane strain and plane stress states is proposed, in the context of weighted integral stochastic finite element method. To derive the formulation, the elements of constitutive matrix are closely investigated in each plane strain and plane stress states and a general mathematical expression, used to represent the spatial randomness (Adhikari and Manohar, 1999; Chakraborty and Bhattacharyya, 2002; Choi and Noh, 1996, 2000; Deodatis and Shinozuka, 1989; Deodatis et al., 1991; Kleiber and Hein, 1992; Shinozuka and Deodatis, 1988), is employed for the spatial uncertainty in the Poisson's ratio. Then the binomial theorem is employed to determine the coefficients of power stochastic field function, which makes it possible for the constitutive matrix to be stochastically decomposed into series of sub-matrices. With the stochastic decomposition of constitutive matrices, it becomes possible to express the element stiffness matrix as a function of random variables defined as a weighted integral of power stochastic field function over the domain of finite elements. The uncertain Poisson's ratio is assumed to follow the Gaussian distribution and the general formula for n-th joint moment (Lin, 1967), which is valid for Gaussian random variables, is employed.

2. Constitutive matrix

The constitutive matrices for plain stress (pss) and plane strain (psn) states are as follows:

$$\mathbf{D}_{\text{pss}} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}, \quad \mathbf{D}_{\text{psn}} = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} \tag{1}$$

Furthermore, in case of plane strain state, the constitutive matrix can be rearranged as

$$\mathbf{D}_{\text{psn}} = \frac{E}{3(1+v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} + \frac{2E}{3(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} = \mathbf{D}_{(a)} + \mathbf{D}_{(b)} \tag{2}$$

Through the investigation on Eqs. (1) and (2), it is noted that the elements of constitutive matrix is consists of the combination of following fraction forms:

$$\frac{1}{1 \pm x} \quad \text{and} \quad \frac{x}{1 \pm x} \tag{3}$$

In the elementary mathematics, it is well known that the following polynomial expansion is satisfied if the range of variable is $|x| < 1.0$:

$$(x)\frac{1}{1 \pm x} = (x)(1 \mp x + x^2 \mp x^3 + \cdots) \tag{4}$$

With Eqs. (1), (2) and (4), it is possible to establish an alternative way in expressing the constitutive matrix, i.e., in an expansion form.

2.1. Stochastic expansion of constitutive matrix

A simple mathematical expression for the spatial randomness in a certain system parameter is $S(\mathbf{x}) = S_o[1 + f_s(\mathbf{x})]$ (Adhikari and Manohar, 1999; Chakraborty and Bhattacharyya, 2002; Choi and Noh, 1996; Deodatis and Shinozuka, 1989; Deodatis et al., 1991; Kleiber and Hein, 1992; Shinozuka and Deodatis, 1988), where S_o is the mean value of S and $f_s(\mathbf{x})$ is a two dimensional homogeneous stochastic field function with zero mean. Following this, the Poisson's ratio, which is assumed to have spatial uncertainty, can be expressed as follows:

$$v(\mathbf{x}) = v_o[1 + f_v(\mathbf{x})] \tag{5}$$

where $f_v(\mathbf{x})$ is a stochastic field function representing the spatial randomness in Poisson's ratio v and $\mathbf{x}$ is a spatial position vector belongs to the domain of structure.

Therefore, replacing (5) into (1) or (2) and employing expansion as given in Eq. (4), each element of constitutive matrix can be expressed in an alternative way as a function of stochastic field function $f(x)$, i.e., in a stochastically decomposed form (see Appendix A). The indirect verification of the use of Eq. (4) with substitution of Eq. (5) is given in Appendix B.

2.2. Plane stress state

Employing the results as given in Appendix A, the constitutive matrix for plane stress state can be written as given in Eq. (6)

$$\mathbf{D}_{\text{pss}} \approx E \begin{bmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & 0 \\ 0 & 0 & e_{33} \end{bmatrix} \tag{6}$$

where the elements e_{ij}s are

$$\begin{Bmatrix} e_{11} = e_{22} \\ e_{12} = e_{21} \\ e_{33} \end{Bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} 1 \\ f(\mathbf{x}) \\ f^2(\mathbf{x}) \\ f^3(\mathbf{x}) \end{Bmatrix} \tag{7}$$

The constants in (7) are evaluated as follows (see Appendix A):

$$\begin{aligned} &\alpha_0 = 1 + \sum_{k=1}^{\infty} \binom{k}{0} v_o^{2k}; \quad \alpha_i = \sum_{k=1,k\geqslant i}^{\infty} \binom{k}{i} v_o^{2k}, \quad i = 1,2,3 \\ &\beta_i = \sum_{k=1,2k-1\geqslant i}^{\infty} \binom{2k-1}{i} v_o^{2k-1}, \quad i = 0,1,2,3 \\ &\gamma_0 = \frac{1}{2}\left\{1 + \sum_{k=1}^{\infty} (-1)^k \binom{k}{0} v_o^k\right\}; \quad \gamma_i = \frac{1}{2} \sum_{k=1,k\geqslant i}^{\infty} (-1)^k \binom{k}{i} v_o^k, \quad i = 1,2,3 \end{aligned} \tag{8}$$

where, $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$.

2.3. Plane strain state

As given in Eq. (2), the constitutive matrix for plane strain has to be divided into two parts mathematically for the application of expansion formula of Eq. (4), which leads to the following:

$$\mathbf{D}_{(a)} \approx E \begin{bmatrix} e_{(a)11} & e_{(a)12} & 0 \\ e_{(a)21} & e_{(a)22} & 0 \\ 0 & 0 & e_{(a)33} \end{bmatrix}, \quad \mathbf{D}_{(b)} \approx E \begin{bmatrix} e_{(b)11} & e_{(b)12} & 0 \\ e_{(b)21} & e_{(b)22} & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \tag{9}$$

where the elements $e_{(a)ij}$ and $e_{(b)ij}$ are

$$\begin{Bmatrix} e_{(a)11} = e_{(a)22} \\ e_{(a)21} = e_{(a)12} \\ e_{(a)33} \\ e_{(b)11} = e_{(b)22} \\ e_{(b)12} = e_{(b)21} \end{Bmatrix} = \begin{bmatrix} \kappa_0 & \kappa_1 & \kappa_2 & \kappa_3 \\ \theta_0 & \theta_1 & \theta_2 & \theta_3 \\ \rho_0 & \rho_1 & \rho_2 & \rho_3 \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 \\ \varepsilon_0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix} \begin{Bmatrix} 1 \\ f(\mathbf{x}) \\ f^2(\mathbf{x}) \\ f^3(\mathbf{x}) \end{Bmatrix} \tag{10}$$

The constants in (10) are evaluated as follows (see also Appendix A):

$$\begin{aligned} &\kappa_0 = \frac{1}{3}\left\{1 + \sum_{k=1}^{\infty} 2(-1)^k \binom{k}{0} v_o^k\right\}; \quad \kappa_i = \frac{1}{3} \sum_{k=1,k\geqslant i}^{\infty} 2(-1)^k \binom{k}{i} v_o^k, \quad i = 1,2,3 \\ &\theta_i = \frac{1}{3} \sum_{k=1,k\geqslant i}^{\infty} (-1)^{k+1} \binom{k}{i} v_o^k, \quad i = 0,1,2,3 \\ &\rho_0 = \frac{1}{6}\left\{1 + \sum_{k=1}^{\infty} 3(-1)^k \binom{k}{0} v_o^k\right\} \quad \rho_i = \frac{1}{2} \sum_{k=1,k\geqslant i}^{\infty} (-1)^k \binom{k}{i} v_o^k, \quad i = 1,2,3 \end{aligned} \tag{11a}$$

and

$$\delta_0 = \frac{2}{3}\left\{1 + \sum_{k=1}^{\infty} 2^{k-1}\binom{k}{0} v_o^k\right\}; \quad \delta_i = \frac{2}{3}\sum_{k=1,k\geqslant i}^{\infty} 2^{k-1}\binom{k}{i} v_o^k, \quad i = 1,2,3$$
$$\varepsilon_i = \frac{2}{3}\sum_{k=1}^{\infty} 2^{k-1}\binom{k}{i} v_o^k, \quad i = 0,1,2,3 \tag{11b}$$

And finally, the constitutive matrix for plane strain state becomes

$$\mathbf{D}_{\mathrm{psn}} = \mathbf{D}_{(a)} + \mathbf{D}_{(b)} \approx E\begin{bmatrix} e_{(a)11} + e_{(b)11} & e_{(a)12} + e_{(b)12} & 0 \\ e_{(a)21} + e_{(b)21} & e_{(a)22} + e_{(b)22} & 0 \\ 0 & 0 & e_{(a)33} + 1/3 \end{bmatrix} \tag{12}$$

As seen in Eqs. (7) and (10), terms with higher order than $f^3(x)$ is assumed small enough to be neglected. As widely noticed, the stochastic field function lies within the range of $-1 + \eta_f < f(x) < 1 - \eta_f$, where $0 < \eta_f < 1$.

2.4. Convergence of expansion coefficients

The accuracy and efficiency of the proposed formulation depend on the convergence characteristics of the coefficients in Eqs. (8) and (11). Therefore the expansion order k in order for the coefficients in Eqs. (8) and (11) to converge is examined. As seen in Figs. 1 and 2, the highest order is around 35 for δ_3 (for

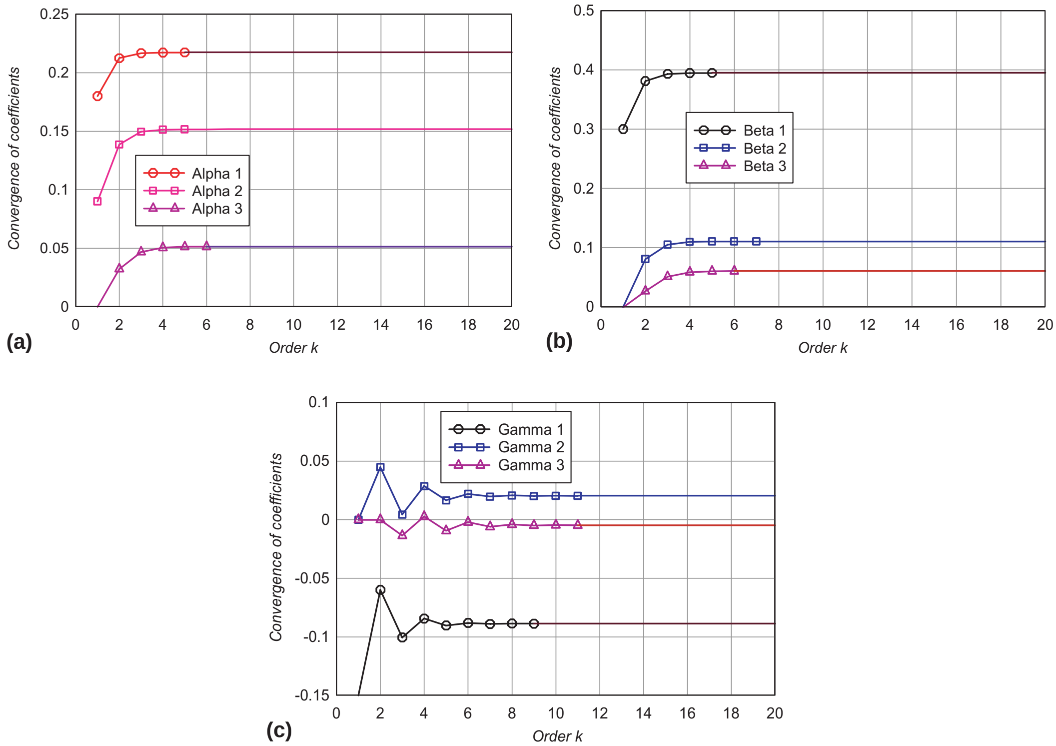

Fig. 1. Convergence of expansion coefficients for plane stress state. (a) For α_i, (b) for β_i, (c) for γ_i.

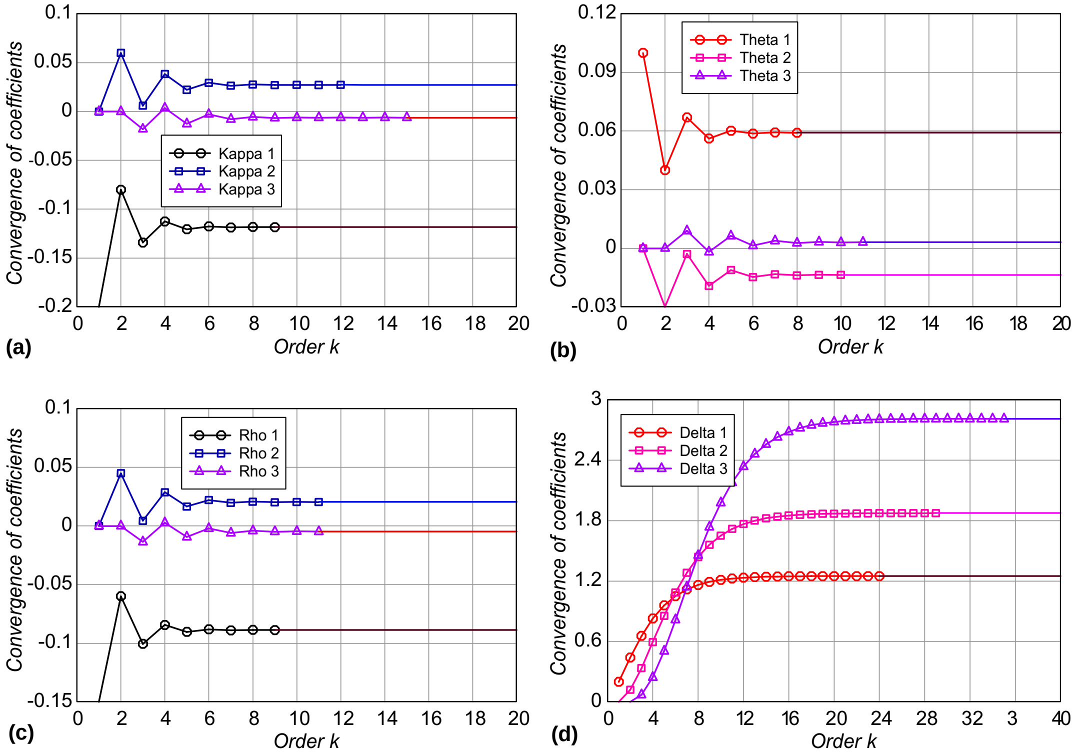

Fig. 2. Convergence of expansion coefficients for plane strain state. (a) For κ_i, (b) for θ_i, (c) for ρ_i, (d) for δ_i.

ε_3 also), and smaller than 11 for the rest of coefficients. The effort for convergence is less for plane stress than for plane strain. The symbols are used to denote the convergence in the order of 10^{-4}. The results shown in Figs. 1 and 2 are the values of coefficients themselves and evaluated with $v_o = 0.3$.

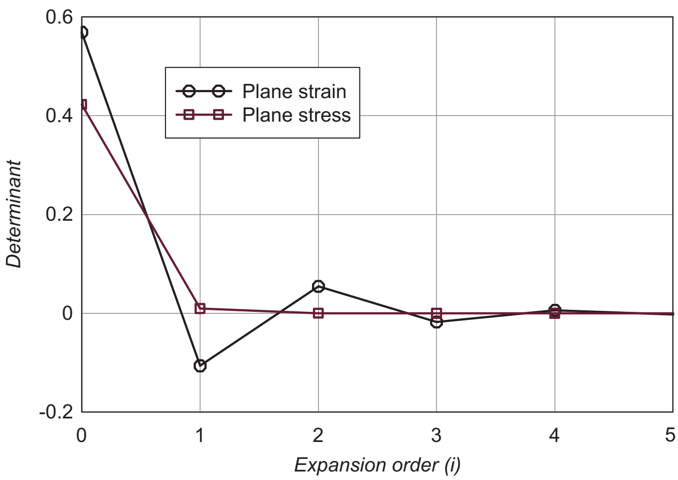

Fig. 3. Determinant of series of sub-matrices $\mathbf{D}^{(i)}$.

Fig. 3 illustrates the series of determinant of sub-matrices $\mathbf{D}^{(i)}$ (excluding elastic modulus E) to show indirectly the anticipated influences of respective sub-matrices as the order (i) is increased. The determinants are evaluated with $\nu_o = 0.3$. As can be noted in the figure, the determinant of sub-matrices is reduced considerably as the order (i) is increased. This implies that the contributions from the higher order sub-matrices are to be very small. Referring to Eqs. (6) and (12), the determinants of series of sub-matrices for plane stress and plane strain states are evaluated as follows:

$$\begin{aligned} |\mathbf{D}^{(i)}_{\text{pss}}| &= (\alpha_i^2 - \beta_i^2)\gamma_i \\ |\mathbf{D}^{(i)}_{\text{psn}}| &= [(\kappa_i + \delta_i)^2 - (\theta_i + \varepsilon_i)^2]\rho_i \end{aligned} \tag{13}$$

3. Stochastic element stiffness matrix

3.1. Mean and deviatoric stiffness

As a result of foregoing contents, for either plane stress and plane strain states, the constitutive matrix is decomposed as follows:

$$\mathbf{D} \approx \mathbf{D}^{(0)} + f(\mathbf{x})\mathbf{D}^{(1)} + f^2(\mathbf{x})\mathbf{D}^{(2)} + f^3(\mathbf{x})\mathbf{D}^{(3)} \tag{14}$$

where $\mathbf{D}^{(i)} = E\begin{bmatrix} \alpha_i & \beta_i & 0 \\ \beta_i & \alpha_i & 0 \\ 0 & 0 & \gamma_i \end{bmatrix}$ for plane stress and $\mathbf{D}^{(i)} = E\begin{bmatrix} \kappa_i + \delta_i & \theta_i + \varepsilon_i & 0 \\ \theta_i + \varepsilon_i & \kappa_i + \delta_i & 0 \\ 0 & 0 & \rho_i \end{bmatrix}$ for plane strain ($i = 1,2,3$) and $\mathbf{D}^{(0)}$ is the same as the original deterministic constitutive matrix with mean of Poisson's ratio ν_o.

With the aid of strain–displacement matrix $\mathbf{B}$, the element stiffness in the finite element method, having element domain of Ω^e, is constructed as follows:

$$\mathbf{k}^e = \int_{\Omega^e} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}\Omega^e \tag{15}$$

Substituting (14) into (15), the element stiffness matrix $\mathbf{k}^e$ is written as the sum of deterministic and deviatoric stiffness as

$$\mathbf{k}^e = \int_{\Omega^e} \sum_{i=0}^{3} f^i(\mathbf{x})\mathbf{B}^{\mathrm{T}}\mathbf{D}^{(i)}\mathbf{B}\,\mathrm{d}\Omega^e = \mathbf{k}_{\text{det}} + \Delta\mathbf{k}^e \tag{16}$$

where considering the relation $E[f(\mathbf{x}_1)f(\mathbf{x}_2)] = R_{ff}(\boldsymbol{\xi}_e = \mathbf{x}_2 - \mathbf{x}_1)$, the mean stiffness is

$$\begin{aligned} E[\mathbf{k}^e] &= \mathbf{k}^e_o = \mathbf{k}_{\text{det}} + \Delta\mathbf{k}^{e(2)}_o \\ \mathbf{k}_{\text{det}} &= \int_{\Omega^e} \mathbf{B}^{\mathrm{T}}\mathbf{D}^{(0)}\mathbf{B}\,\mathrm{d}\Omega^e \\ \Delta\mathbf{k}^{e(2)}_o &= \int_{\Omega^e} R_{ff}(\boldsymbol{\xi}_e)\mathbf{B}^{\mathrm{T}}\mathbf{D}^{(2)}\mathbf{B}\,\mathrm{d}\Omega^e \end{aligned} \tag{17}$$

and

$$\Delta\mathbf{k}^e = \mathbf{k}^e - \mathbf{k}^e_o = \Delta\mathbf{k}^{e(1)} + \Delta\mathbf{k}^{e(2)} + \Delta\mathbf{k}^{e(3)} - \Delta\mathbf{k}^{e(2)}_o = \Delta\hat{\mathbf{k}}^e - \Delta\mathbf{k}^{e(2)}_o \tag{18}$$

where, subscript 'det' denotes the original deterministic part of element stiffness matrix and $\Delta\mathbf{k}^{e(i)} = \int_{\Omega^e} f^i(\mathbf{x})\mathbf{B}^{\mathrm{T}}\mathbf{D}^{(i)}\mathbf{B}\,\mathrm{d}\Omega^e$.

3.2. Definition of random variable

To demonstrate explicitly the stiffness matrix is to be a function of random variable and to show the random variable itself, it is indispensable to decompose the strain–displacement matrix **B** into the sum of matrix $\mathbf{B}_i$ multiplied by an independent polynomial p_i as Choi and Noh (1996)

$$\mathbf{B} = \sum_{i}^{N_p} \mathbf{B}_i p_i \tag{19}$$

where matrix $\mathbf{B}_i$ has constants as elements and N_p is the number of independent polynomials in strain–displacement matrix **B**. As a consequence, the deviatoric stiffness matrices $\Delta\mathbf{k}^{e(k)}$ $(k = 1, 2, 3)$ in (18) can be written as follows:

$$\Delta\mathbf{k}^{e(k)} = \int_{\Omega^e} f^k(\mathbf{x})\mathbf{B}^{\mathrm{T}}\mathbf{D}^{(k)}\mathbf{B}\,\mathrm{d}\Omega^e = \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} \mathbf{B}_i^{\mathrm{T}}\mathbf{D}^{(k)}\mathbf{B}_j X_{ij}^{(k)} \tag{20}$$

where $X_{ij}^{(k)} = \int_{\Omega^e} f^k(\mathbf{x}) p_i p_j \,\mathrm{d}\Omega^e$ with which random variables in the weighted integral stochastic finite element method are defined.

3.3. Total number of random variables

According to Eqs. (18) and (20), the total number of random variable, N_{RV}, for each finite element is evaluated as

$$\begin{aligned} N_{\mathrm{RV}} &= \left[\frac{1}{2}N_p(N_p+1)\right] \times 3 \\ &= \frac{3}{2}N_p(N_p+1) \quad \text{or} \\ &= \left[\frac{1}{2}N_p(N_p+1)\right]_{\text{for (1),(2) and (3)}} \\ &= N_{\mathrm{RV}^{(1)}} + N_{\mathrm{RV}^{(2)}} + N_{\mathrm{RV}^{(3)}} \end{aligned} \tag{21}$$

Therefore the random variables in total, for all the finite elements in the domain, can be written in two different ways as

$$\begin{aligned} \{X\}_{\mathrm{RV}=1\cdots N_{\mathrm{RV}}}^{e=1\cdots N_e} &= \left\langle X_1^1, X_2^1, \ldots, X_{N_{\mathrm{RV}}}^1, \quad X_1^2, X_2^2, \ldots, X_{N_{\mathrm{RV}}}^2, \quad \ldots, \quad X_1^{N_e}, X_2^{N_e}, \ldots, X_{N_{\mathrm{RV}}}^{N_e} \right\rangle^{\mathrm{T}} \\ \{X\}_{\mathrm{RV}^{(i)};i=1,2,3}^{e=1\cdots N_e} &= \left\langle X_1^1, X_2^2, \ldots, X_{N_{\mathrm{RV}^{(1)}}}^{N_e}, \quad X_1^1, X_2^2, \ldots, X_{N_{\mathrm{RV}^{(2)}}}^{N_e}, \quad \ldots, \quad X_1^1, X_2^2, \ldots, X_{N_{\mathrm{RV}^{(3)}}}^{N_e} \right\rangle^{\mathrm{T}} \end{aligned} \tag{22}$$

where N_e denotes the number of finite elements in the finite element mesh. In the following sections, X_{RV}^e or $X_{\mathrm{RV}^{(i)}}^e$ $(i = 1, 2, 3)$ is used exclusively rather than $X_{ij}^{(k)}$, following the definitions in expressions (21) and (22).

4. Response statistics

The response variability in stochastic FE analysis is generally given by the coefficient of variation, COV. The COV is evaluated as a square root of the ratio of variance of response R, σ_R^2 , to the square of mean response $\bar{R}$.

$$\mathrm{COV} = \left[\frac{\sigma_R^2}{\bar{R}^2}\right]^{\frac{1}{2}} \tag{23}$$

And the cross-correlation for two distinct degree of freedoms i and j is

$$\mathrm{COV}_{ij} = \left[\frac{(R_i - \bar{R}_i)(R_j - \bar{R}_j)}{\bar{R}_i \bar{R}_j}\right]^{\frac{1}{2}} \tag{24}$$

4.1. Mean centered series expansion of response vector

In the previous section it is noted that $\Delta\mathbf{k}^e$ is given as a function of random variable X^e_{RV}, which is in the form of weighted integral of power stochastic field function. As a consequence, not only the element stiffness matrix but also the assembled global stiffness is given as functions of random variable X^e_{RV}. In addition, owing to the fact that the response vector $\mathbf{U}$ is obtained as an inversion of stiffness matrix multiplied by the deterministic force vector $\mathbf{P}$, vector $\mathbf{U}$ is a function of X^e_{RV}, viz.

$$\mathbf{U} = \mathbf{U}(X^e_{\mathrm{RV}}) \tag{25}$$

This deduction enables us to perform mean centered series expansion of response vector $\mathbf{U}$ with respect to the random variable X^e_{RV} and then to obtain the mean and covariance of response in the sequel.

The first-order expansion of displacement vector $\mathbf{U}$ with respect to the mean of random variable is as follows:

$$\mathbf{U} \approx \mathbf{U}_o + \sum_{e=1}^{N_e} \sum_{\mathrm{RV}=1}^{N_{\mathrm{RV}}} \left(X^e_{\mathrm{RV}} - X^{eo}_{\mathrm{RV}}\right) \left[\frac{\partial \mathbf{U}}{\partial X^e_{\mathrm{RV}}}\right]_E \tag{26}$$

where, $\mathbf{U}_o$ is a displacement vector evaluated with mean stiffness matrix, (17), and superscript o in the random variable and subscript E outside the bracket denote mean value of random variable and the evaluation at the mean, respectively. Here, it must be noted that in the stochastic stiffness matrix, three kinds of random variables are involved, viz. $X_{\mathrm{RV}^{(1)}}$, which contains $f(\mathbf{x})$ as an integrand and $X_{\mathrm{RV}^{(2)}}$, containing $f^2(\mathbf{x})$, and $X_{\mathrm{RV}^{(3)}}$, containing $f^3(\mathbf{x})$, as seen in Eq. (18). Therefore, it becomes possible for Eq. (26) to be transformed into Eq. (27) as follows:

$$\begin{aligned}
\mathbf{U} \approx{} & \mathbf{U}_o - \sum_{e=1}^{N_e} \sum_{\mathrm{RV}^{(1)}=1}^{N_{\mathrm{RV}(1)}} X^e_{\mathrm{RV}^{(1)}} \mathbf{K}_o^{-1} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}^{(1)}}}\right]_E \mathbf{U}_o - \sum_{e=1}^{N_e} \sum_{\mathrm{RV}^{(2)}=1}^{N_{\mathrm{RV}(2)}} (X^e_{\mathrm{RV}^{(2)}} - X^{eo}_{\mathrm{RV}^{(2)}}) \mathbf{K}_o^{-1} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}^{(2)}}}\right]_E \mathbf{U}_o \\
& - \sum_{e=1}^{N_e} \sum_{\mathrm{RV}^{(3)}=1}^{N_{\mathrm{RV}(3)}} X^e_{\mathrm{RV}^{(3)}} \mathbf{K}_o^{-1} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}^{(3)}}}\right]_E \mathbf{U}_o \\
={} & \mathbf{U}_o - \sum_{e=1}^{N_e} \sum_{\mathrm{RV}=1}^{N_{\mathrm{RV}}} X^e_{\mathrm{RV}} \mathbf{K}_o^{-1} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}}}\right]_E \mathbf{U}_o + \sum_{e=1}^{N_e} \sum_{\mathrm{RV}^{(2)}=1}^{N_{\mathrm{RV}(2)}} \mathbf{K}_o^{-1} X^{eo}_{\mathrm{RV}^{(2)}} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}^{(2)}}}\right]_E \mathbf{U}_o
\end{aligned} \tag{27}$$

In the transformation from (26), (27), considering the partial differentiation of the equilibrium equation, it is noted that (Choi and Noh, 1996, 2000; Deodatis et al., 1991)

$$\left[\frac{\partial \mathbf{U}}{\partial X^e_{\mathrm{RV}}}\right]_E = -\mathbf{K}_o^{-1} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}}}\right]_E \mathbf{U}_o \tag{28}$$

and

$$
\begin{aligned}
X^{eo}_{\mathrm{RV}^{(1)}} &= \int_{\Omega^e} E[f(\mathbf{x}_e)] p_i p_j \, \mathrm{d}\Omega^e = 0 \\
X^{eo}_{\mathrm{RV}^{(3)}} &= \int_{\Omega^e} E\left[f^3(\mathbf{x}_e)\right] p_i p_j \, \mathrm{d}\Omega^e = 0 \\
X^{eo}_{\mathrm{RV}^{(2)}} &= \int_{\Omega^e} E\left[f^2(\mathbf{x}_e)\right] p_i p_j \, \mathrm{d}\Omega^e = \int_{\Omega^e} R_{ff}(\boldsymbol{\xi}_e) p_i p_j \, \mathrm{d}\Omega^e
\end{aligned} \tag{29}
$$

The relative distance vector $\boldsymbol{\xi}_e$ is defined in terms of the position vector $\mathbf{x}$, which belongs to the domain of finite element under consideration.

4.2. Response statistics

The mean of response is evaluated with expectation operation $E[\bullet]$ in Eq. (27). In this operation, since the expectation of random variables $X^e_{\mathrm{RV}^{(1)}}, X^e_{\mathrm{RV}^{(3)}}$ vanishes as noted in Eq. (29), the mean of response is obtained as

$$E[\mathbf{U}] \approx \mathbf{U}_o \tag{30}$$

The covariance of response can be evaluated as follows:

$$\mathrm{Cov}[\mathbf{U},\mathbf{U}] = E[\Delta\mathbf{U}\Delta\mathbf{U}^{\mathrm{T}}] \tag{31}$$

where

$$\Delta\mathbf{U} = \mathbf{U} - E[\mathbf{U}] = -\sum_{e=1}^{N_e} \sum_{\mathrm{RV}=1}^{N_{\mathrm{RV}}} X^e_{\mathrm{RV}} \mathbf{K}_o^{-1} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}}}\right]_E \mathbf{U}_o + \sum_{e=1}^{N_e} \sum_{\mathrm{RV}^{(2)}=1}^{N_{\mathrm{RV}(2)}} \mathbf{K}_o^{-1} X^{eo}_{\mathrm{RV}^{(2)}} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}^{(2)}}}\right]_E \mathbf{U}_o \tag{32}$$

The second term in Eq. (32) is a constant one as it is already evaluated in the expansion procedure in Eq. (27). Designating two double summation terms in (32) as Ξ_A and $\bar{\Xi}_{(2)}$ for simplicity, the formula for covariance is given as follows:

$$
\begin{aligned}
\mathrm{Cov}[\mathbf{U},\mathbf{U}] &= E[\Delta\mathbf{U}\Delta\mathbf{U}^{\mathrm{T}}] \\
&= E\left[(-\Xi_A + \bar{\Xi}_{(2)})(-\Xi_A + \bar{\Xi}_{(2)})^{\mathrm{T}}\right] \\
&= E\left[\Xi_A \Xi_A^{\mathrm{T}}\right] - \bar{\Xi}_{(2)} \bar{\Xi}_{(2)}^{\mathrm{T}} (\because \bar{\Xi}_{(1)} = 0, \bar{\Xi}_{(3)} = 0)
\end{aligned} \tag{33}
$$

And each term in (33) can be evaluated as follows:

$$
\begin{aligned}
E[\Xi_A \Xi_A^{\mathrm{T}}] &= E\left[\left(\sum_{ei=1}^{N_e} \sum_{\mathrm{RV}i=1}^{N_{\mathrm{RV}}} X^{ei}_{\mathrm{RV}i} \mathbf{K}_o^{-1} \left[\frac{\partial \mathbf{K}}{\partial X^{ei}_{\mathrm{RV}i}}\right]_E \mathbf{U}_o\right) \left(\sum_{ej=1}^{N_e} \sum_{\mathrm{RV}j=1}^{N_{\mathrm{RV}}} X^{ej}_{\mathrm{RV}j} \mathbf{K}_o^{-1} \left[\frac{\partial \mathbf{K}}{\partial X^{ej}_{\mathrm{RV}j}}\right]_E \mathbf{U}_o\right)^{\mathrm{T}}\right] \\
&= \sum_{ei=1}^{N_e} \sum_{ej=1}^{N_e} \sum_{\mathrm{RV}i=1}^{N_{\mathrm{RV}}} \sum_{\mathrm{RV}j=1}^{N_{\mathrm{RV}}} \mathbf{K}_o^{-1} \left[\frac{\partial \mathbf{K}}{\partial X^{ei}_{\mathrm{RV}i}}\right]_E \mathbf{U}_o \mathbf{U}_o^{\mathrm{T}} \left[\frac{\partial \mathbf{K}}{\partial X^{ej}_{\mathrm{RV}j}}\right]_E^{\mathrm{T}} \mathbf{K}_o^{-\mathrm{T}} E[X^{ei}_{\mathrm{RV}i} X^{ej}_{\mathrm{RV}j}] \\
&= \sum_{ei,ej=1}^{N_e} \mathbf{K}_o^{-1} \bar{\bar{\mathbf{F}}}_{eiej,E} \mathbf{K}_o^{-\mathrm{T}}
\end{aligned} \tag{34a}
$$

$$\bar{\Xi}_{(2)} \bar{\Xi}_{(2)}^{\mathrm{T}} = \left(\sum_{e=1}^{N_e} \sum_{\mathrm{RV}^{(2)}=1}^{N_{\mathrm{RV}(2)}} \mathbf{K}_o^{-1} X^{eo}_{\mathrm{RV}^{(2)}} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}^{(2)}}}\right]_E \mathbf{U}_o\right) \left(\sum_{e=1}^{N_e} \sum_{\mathrm{RV}^{(2)}=1}^{N_{\mathrm{RV}(2)}} \mathbf{K}_o^{-1} X^{eo}_{\mathrm{RV}^{(2)}} \left[\frac{\partial \mathbf{K}}{\partial X^e_{\mathrm{RV}^{(2)}}}\right]_E \mathbf{U}_o\right)^{\mathrm{T}} \tag{34b}$$

After some mathematical manipulation, it is noted in Eq. (34a) that

$$\bar{\bar{\mathbf{F}}}_{eiej,E} = E[\Delta\hat{\mathbf{k}}^{ei}\mathbf{U}_o\mathbf{U}_o^{\mathrm{T}}\Delta\hat{\mathbf{k}}^{ej}] \tag{35}$$

As $\Delta\hat{\mathbf{k}}^{ei}\mathbf{U}_o$ in Eq. (35) has force-equivalent quantity, $\bar{\bar{\mathbf{F}}}_{eiej,E}$ can be termed as "force-equivalent covariance matrix". Therefore the essential part in obtaining the covariance of response is the evaluation of force-equivalent covariance matrix of Eq. (35).

4.3. Evaluation of force-equivalent covariance matrix

Since the deviatoric stiffness matrices are given as

$$\begin{aligned} \Delta\hat{\mathbf{k}}^{e} &= \sum_{k=1}^{3}\Delta\mathbf{k}^{e(k)} \\ \Delta\mathbf{k}^{e(k)} &= \int_{\Omega^e} f^k(\mathbf{x})\mathbf{B}^{\mathrm{T}}\mathbf{D}^{(k)}\mathbf{B}\,\mathrm{d}\Omega^e; \quad k = 1,2,3 \end{aligned} \tag{36}$$

the following is satisfied for two distinct finite elements *ei* and *ej*

$$\begin{aligned} \bar{\bar{\mathbf{F}}}_{eiej,E} &= E\left[\left(\Delta\mathbf{k}^{ei(1)} + \Delta\mathbf{k}^{ei(2)} + \Delta\mathbf{k}^{ei(3)}\right)\mathbf{U}_o\mathbf{U}_o^{\mathrm{T}}\left(\Delta\mathbf{k}^{ej(1)} + \Delta\mathbf{k}^{ej(2)} + \Delta\mathbf{k}^{ej(3)}\right)\right] \\ &= \int_{\Omega^{ei}}\int_{\Omega^{ej}}\sum_{k=1}^{3}\sum_{l=1}^{3}\left\{E\left[f^k(\mathbf{x}_{ei})f^l(\mathbf{x}_{ej})\right]\tilde{\mathbf{k}}^{ei(k)}\mathbf{U}_o\mathbf{U}_o^{\mathrm{T}}\tilde{\mathbf{k}}^{ej(l)}\right\}\mathrm{d}\Omega^{ej}\,\mathrm{d}\Omega^{ei} \end{aligned} \tag{37}$$

where $\tilde{\mathbf{k}}^{ei(k)} = \mathbf{B}_{ei}^{\mathrm{T}}\mathbf{D}_{ei}^{(k)}\mathbf{B}_{ei}$.

If the relationship between expectation on power stochastic field function and auto-correlation function is established, Eq. (35) or Eq. (37) can be rewritten in the following form of equation, which shows explicitly the independent contributions of three distinct deviatoric stiffness terms on the response variability.

$$\bar{\bar{\mathbf{F}}}_{eiej,E} = \sum_{k=1}^{3}\sum_{l=1}^{3}\int_{\Omega^{ei}}\int_{\Omega^{ej}}\hat{R}_f^{(kl)}(\boldsymbol{\xi}_{ij},\boldsymbol{\xi}_{ii},\boldsymbol{\xi}_{jj})\tilde{\mathbf{k}}^{ei(k)}\mathbf{U}_o\mathbf{U}_o^{\mathrm{T}}\tilde{\mathbf{k}}^{ej(l)}\,\mathrm{d}\Omega^{ej}\,\mathrm{d}\Omega^{ei} \tag{38}$$

In Eq. (38), the modified auto-correlation functions $\hat{R}_f^{(kl)}(\boldsymbol{\xi}_{ij},\boldsymbol{\xi}_{ii},\boldsymbol{\xi}_{jj})$ are as follows:

$$\begin{aligned} \hat{R}_f^{(11)}(\boldsymbol{\xi}_{ij},\boldsymbol{\xi}_{ii},\boldsymbol{\xi}_{jj}) &= R_f(\boldsymbol{\xi}_{ij}) \\ \hat{R}_f^{(22)}(\boldsymbol{\xi}_{ij},\boldsymbol{\xi}_{ii},\boldsymbol{\xi}_{jj}) &= 2R_f^2(\boldsymbol{\xi}_{ij}) + R_f(\boldsymbol{\xi}_{ii})R_f(\boldsymbol{\xi}_{jj}) \\ \hat{R}_f^{(33)}(\boldsymbol{\xi}_{ij},\boldsymbol{\xi}_{ii},\boldsymbol{\xi}_{jj}) &= 6R_f^3(\boldsymbol{\xi}_{ij}) + 9R_f(\boldsymbol{\xi}_{ii})R_f(\boldsymbol{\xi}_{jj})R_f(\boldsymbol{\xi}_{ij}) \\ \hat{R}_f^{(13)}(\boldsymbol{\xi}_{ij},\boldsymbol{\xi}_{ii},\boldsymbol{\xi}_{jj}) &= 3R_f(\boldsymbol{\xi}_{jj})R_f(\boldsymbol{\xi}_{ij}) \\ \hat{R}_f^{(31)}(\boldsymbol{\xi}_{ij},\boldsymbol{\xi}_{ii},\boldsymbol{\xi}_{jj}) &= 3R_f(\boldsymbol{\xi}_{ii})R_f(\boldsymbol{\xi}_{ij}) \end{aligned} \tag{39}$$

In deriving Eq. (39), the general formula for *n*-th joint moment, Eq. (40), is employed (Lin, 1967).

$$E[X_1X_2\cdots X_n] = \sum_{k,k\neq j}E[X_{r_1}X_{r_2}\cdots X_{r_{n-2}}]E[X_kX_j] \tag{40}$$

The number of terms N included in the summation is determined as $N = (2m)!/m!2^m$, where $n = 2m$. The function $\hat{R}_f^{(kl)}(\boldsymbol{\xi}_{ij},\boldsymbol{\xi}_{ii},\boldsymbol{\xi}_{jj})$ is zero when "$k+1$" is an odd number. The vector $\boldsymbol{\xi}_{ij}$ denotes inter-element relative distance vector and $\boldsymbol{\xi}_{ii}$ and $\boldsymbol{\xi}_{jj}$ are also relative distance vectors which are defined in each element of *ei* or *ej*. The evaluation of covariance is completed with the substitution of Eq. (38) into Eq. (34a).

5. Numerical examples

To illustrate the performance of the proposed formulation, some example structures such as square plate, simple beam and cantilever are analyzed. The results of the proposed weighted integral (WI) method are compared with those of the classical Monte Carlo simulation (MCS). The sample generation technique used in MCS is a statistical preconditioning with which fairly good statistics can be attained with only a relatively small number of generated sample fields (Yamazaki and Shinozuka, 1990).

The coefficient of variation of stochastic field (σ_{ff}) is assumed to be 0.1 if not mentioned otherwise. For the numerical integration of auto-correlation function $R_{ff}(\xi)$ in Eq. (38), a 10×10 Lobato integration scheme is employed.

The auto-correlation function adopted is as follows:

$$R_{ff}(\boldsymbol{\xi}) = \sigma_{ff}^2 \exp\left\{-\frac{|\xi_1|}{d_1} - \frac{|\xi_2|}{d_2}\right\} \tag{41}$$

where σ_{ff} is the coefficient of variation of stochastic field $f(\mathbf{x})$. The two constants d_1 and d_2 with which the frequency features of stochastic field is represented conceptually are the correlation distances in two orthogonal directions in the plane of structure. In the numerical applications, $d = d_1 = d_2$ is used.

5.1. Example 1: Square in-plane plate (without shear behavior)

The example square plate is shown in Fig. 4. The Young's modulus is $E = 2.1E + 06$, and the thickness of plate is $t = 1.0$. The mean Poisson's ratio v_o is assumed to be 0.20. A pressure load q is applied in the upward direction. In the analysis, 6×6 mesh is used exclusively. Here, all the parameters are given without units so that any units can be specified as long as they are used consistently. In this example, the Poisson effect is highlighted with only a least restraint condition to prevent rigid body motion. The 'COV' of displacement is found at point A.

The COV of displacement as a function of correlation distance d for the state of plane strain is shown in Fig. 5. As seen in the figure, the COV of response in the direction orthogonal to loading exceeds the value of COV of Poisson's ratio, which is assumed to be 0.1. However, in the loading direction, the COV is considerably small showing around 10% of COV of stochastic field. This phenomenon shows clearly not only that the Poisson effect is related to the responses in load-normal direction but also that the randomness in

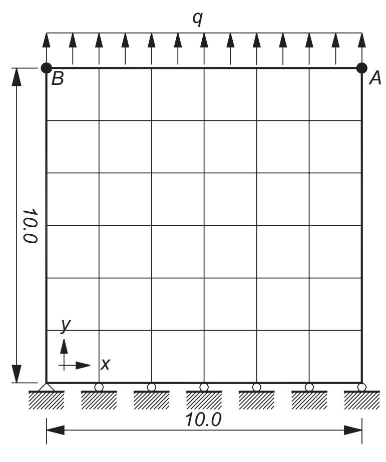

Fig. 4. Example in-plane square plate.

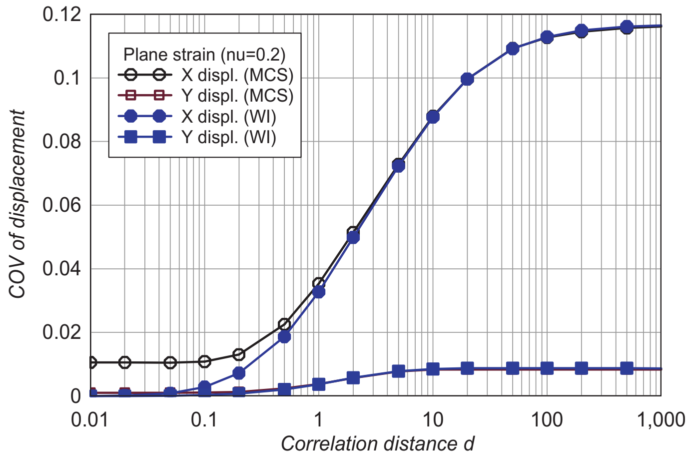

Fig. 5. COV variation in plane strain state.

Poisson's ratio affects the structural response in great extends. In addition, it is noted that the results obtained with the proposed method are in good agreement with those obtained by way of MCS. The plateau region for small value of correlation distance *d* in the MCS comes from the incapability of coarse discrete version of random field in representing the white noise type of stochastic fields.

The response variability for plane stress state is shown in Fig. 6. In this case, the COV of displacement in load-normal direction is revealed to be less than that in the plane strain state and the maximum of response variability is shown to be 0.1. The COV in the response in the loading direction is obtained to be even smaller than the case of plane strain state.

The cross-correlation defined in Eq. (24) for two points *i* and *j*, designated in Fig. 4 as *A* and *B*, is given in Fig. 7. In the evaluation of cross-correlation, a distributed load in *x* direction is employed as an applied load. In this case also, the COV_{ij}'s in the proposed WI are in good agreement with those of MCS.

Fig. 8 compares the distributions of COV in Monte Carlo simulation and proposed weighted integral method to show the global, not just of the point *A* in Fig. 4, similarity in the two analyses. Due to the

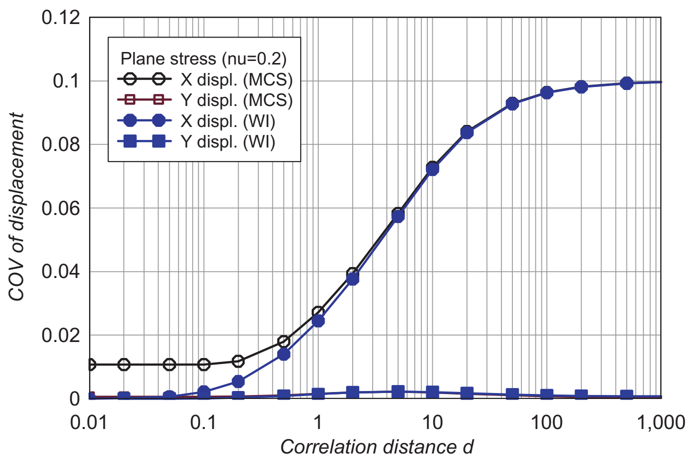

Fig. 6. COV variation in plane stress state.

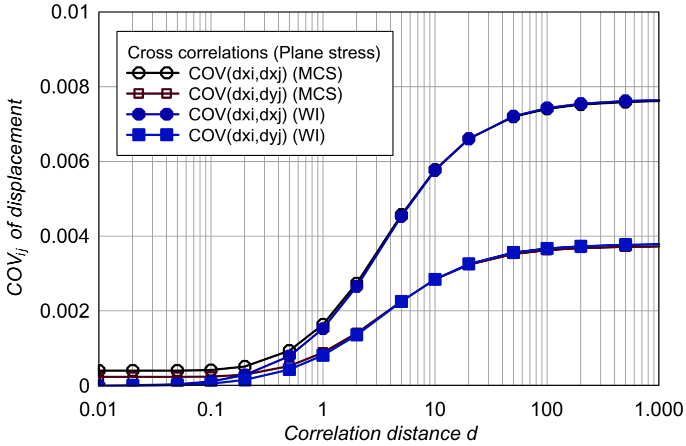

Fig. 7. Cross-correlation COV_{ij} in shear dominant behavior (plane stress state).

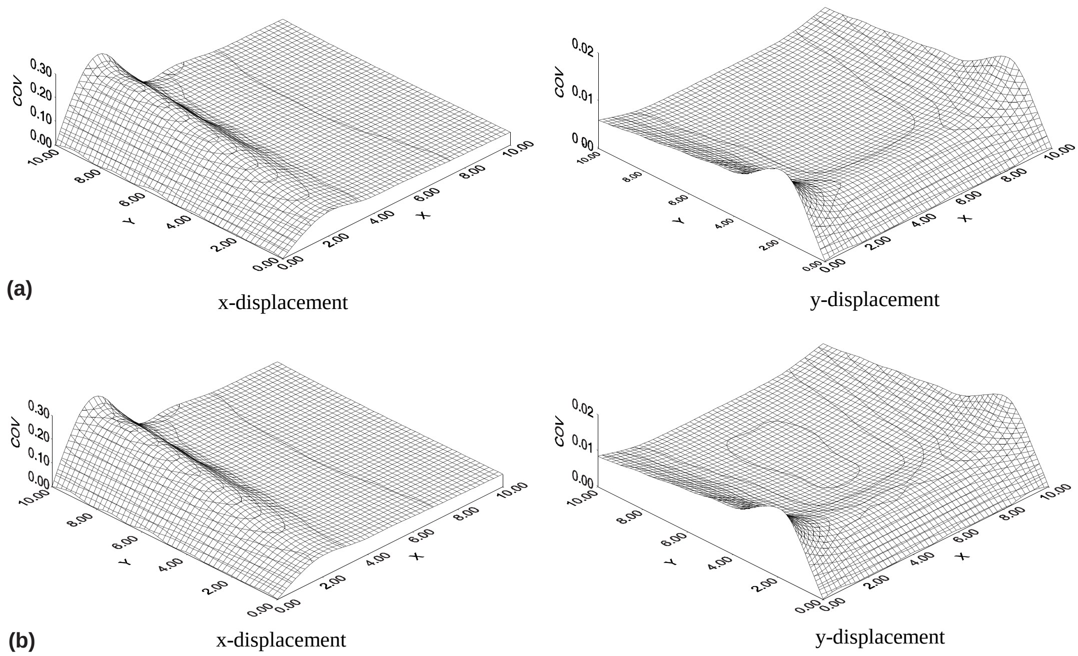

Fig. 8. Comparison of COV for in-plane plate ($d = 50.0$). (a) Monte Carlo simulation, (b) Proposed weighted integral method.

uncertain material parameter, standard deviation of x-displacement for the nodes along the left edge ($x = 0.0$) takes some values. It is apparent, however, that the mean x-displacement in this part is zero, which leads the COV to be infinity. Therefore, the values of COV for x-displacement at this part are forced to be zero when creating the 3D plot in Fig. 8. Though they are not given, the distributions of standard deviation also are examined to show good agreement between MCS and WI.

5.2. Example 2: Simple beam and cantilever (with shear behavior)

In order to investigate the performance of the proposed formulation for structural response, which includes bending as well as shear behavior, beam structures are taken as another example. The material properties are the same as those of foregoing in-plane plate example. The COV s are found at point A in Fig. 9.

The response variability of simple beam and cantilever in the state of plane strain is shown in Figs. 10 and 11 respectively. In this case, the COV appears to be smaller than the case of in-plane plate structure and reaches about 10% of COV of stochastic field. In case of plane stress state the COV is obtained to be less than 5% of COV of stochastic field.

Comparison of the distributions of COV for simple beam example is given in Fig. 12. As is in the in-plane plate example, the similarity over the structural domain in the two analyses is well demonstrated. It has to be noted that the distributions of standard deviation are also in good agreement.

From the results of these two examples, it can be noted that the influence of uncertain Poisson's ratio on the response variability is reduced when bending and shear behaviors are included, and is greater for plane strain state than for plane stress state.

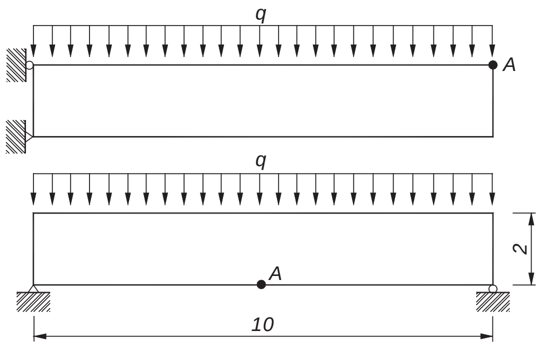

Fig. 9. Example beams.

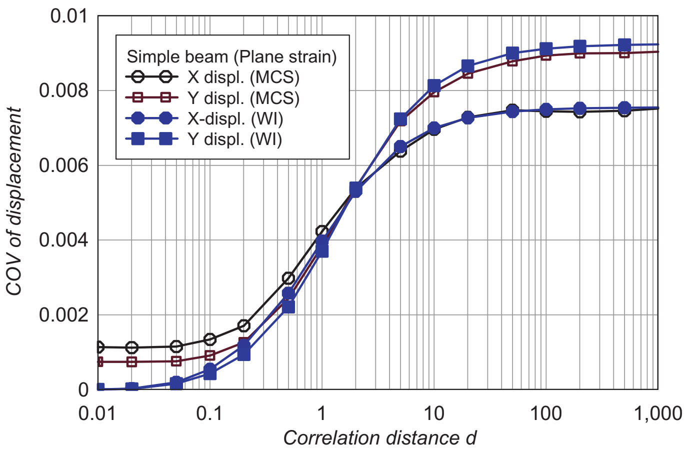

Fig. 10. COV variation of simple beam.

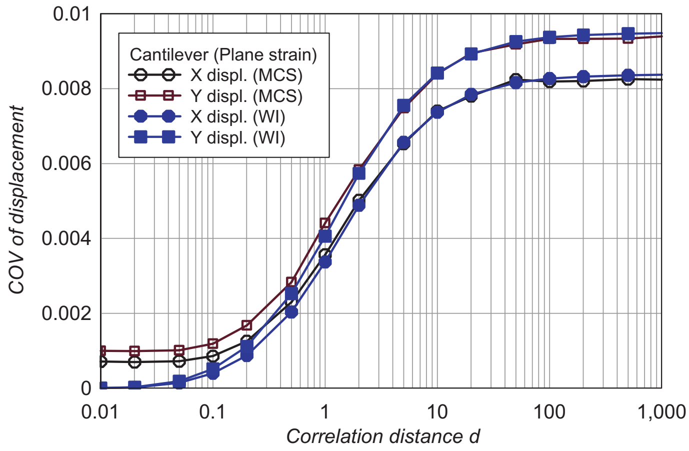

Fig. 11. COV variation of cantilever.

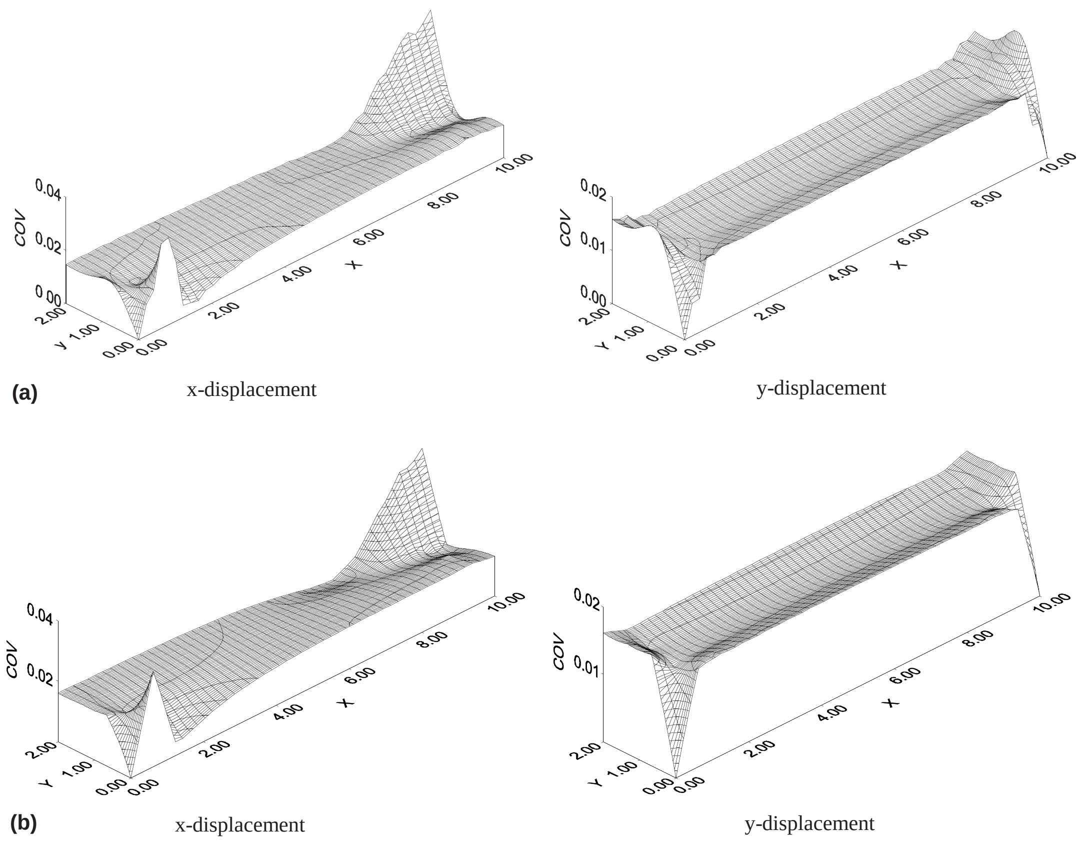

Fig. 12. Comparison of COV for beam example ($d = 2.0$). (a) Monte Carlo simulation, (b) Proposed weighted integral method.

5.3. Effect of varying value of COV of stochastic field

The effect of intensity of stochasticity, i.e., the value of σ_{ff}, of Poisson's ratio is depicted in Fig. 13 for in-plane plate and simple beam. The COV of response increases as the COV of stochastic field is increased. Even though some discrepancies are shown, especially for beam for larger COV of stochastic field, the COVs in WI and MCS show good agreement. The increase of COV of response is investigated to be proportional to the increase in COV of stochastic field in in-plane plate. In beam example, however, the increase in COV of response is greater than that in the COV of stochastic field.

5.4. Effect of value of Poisson's ratio

Since the structural response is a non-linear function of Poisson's ratio, the value of Poisson's ratio itself is expected to affect the response variability, which is not the case of randomness in elastic modulus that is a linear parameter. In Fig. 14 the influences of value of Poisson's ratio on the COV of response are illustrated.

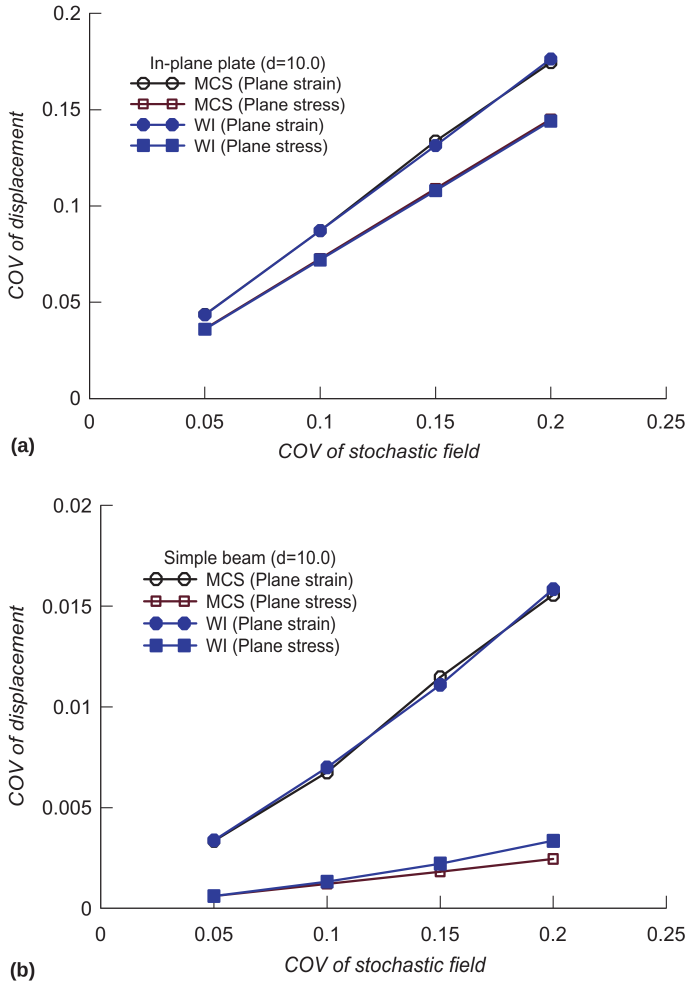

Fig. 13. Effect of varying COV of stochastic field. (a) In-plane square plate ($d = 10.0$), (b) Simple beam ($d = 10.0$).

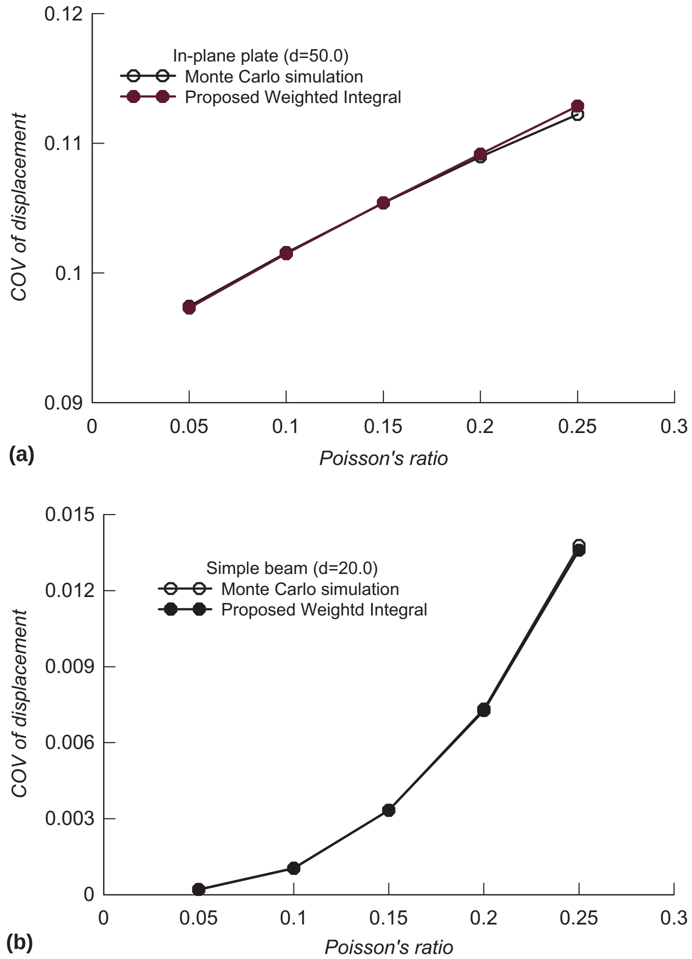

Fig. 14. COV as a function of Poisson's ratio. (a) Plane structure ($d = 50.0$), (b) Simple beam ($d = 20.0$).

As seen in Fig. 14, the COV is not in direct proportion to the value of Poisson's ratio; besides, a non-linear relationship is shown in case of beam structure, which indicates that the structures with bending and shear behavior are more sensitive in the response variability to the value of Poisson's ratio than the structures in axial stress state.

5.5. Brief comments on the application limit

Since the Poisson's ratio has physical constraint of $0 < v < 0.5$, a relation between coefficient of variation of Poisson's ratio $\alpha = \sigma_v/v_o$ and the mean Poisson's ratio v_o must be established. Adopting the peak factor K_p, which determines probability level in the Gaussian distribution, the constraint in the Poisson's ratio becomes $0 < v_o + K_p\sigma_v < 0.5$, where σ_v is a standard deviation of uncertain Poisson's ratio. After some manipulation the following inequality can be established.

$$-1 < K_p\alpha < \left(\frac{v_{\max}}{v_o} - 1\right) \tag{42}$$

Therefore the interval for coefficient of variation α is determined as

$$-\frac{1}{K_p} < \alpha < \frac{(v_{\max}/v_o - 1)}{K_p}, \quad \text{if } K_p > 0 \tag{43a}$$

$$-\frac{1}{K_p} > \alpha > \frac{(v_{\max}/v_o - 1)}{K_p}, \quad \text{if } K_p < 0 \tag{43b}$$

Eq. (43) determines the range of adoptable coefficient of variation of Poisson's ratio when the mean Poisson's ratio is v_o. Eq. (43a) is for right-hand side of v_o and (43b) is for left. Here, it is noted that if the value of mean Poisson's ratio is $v_o = 0.25$ then $(v_{\max}/v_o - 1) = 1.0$. That is, $v_o = 0.25$ is a bifurcation point in Eq. (43). And, since the coefficient of variation is a positive number, only the positive value is a valid one. Accordingly, we take (43a) in the form of $\alpha < \frac{(v_{\max}/v_o - 1)}{K_p}$ when $v_o > 0.25$ and take (43b) as $\alpha < -\frac{1}{K_p}$ when $v_o < 0.25$. These limits are illustrated in Fig. 15 for values of K_p from 1 to 4. The solid triangle in Fig. 15 denotes the state of foregoing example analysis.

Fig. 16 illustrates an example of application limit when K_p is taken as 4.0. Employing Eq. (43), the adoptable maximum coefficient of variation is evaluated as $\alpha < -\frac{1}{K_p} (= 0.25)$ when $v_o = 0.1$ (designated as circle in Fig. 15), thus the standard deviation of Poisson's ratio σ_v becomes 0.025. Similarly, in case of $v_o = 0.4$ (designated as square in Fig. 15), $\alpha < \frac{(v_{\max}/v_o - 1)}{K_p} (= 0.0625)$ leading to the same σ_v of 0.025.

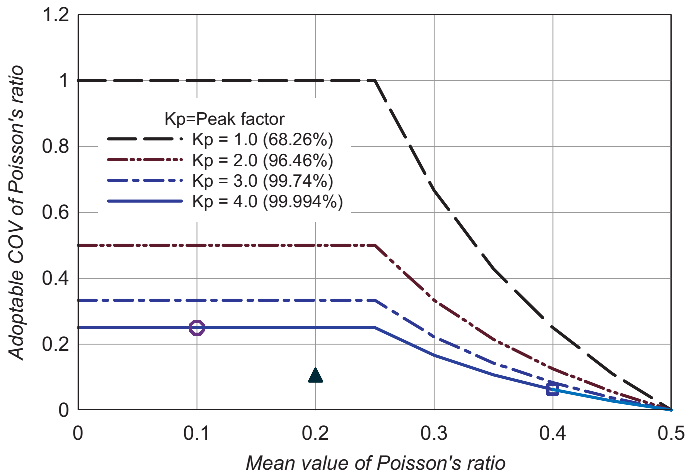

Fig. 15. Application limits: theoretical maximum mean Poisson's ratio.

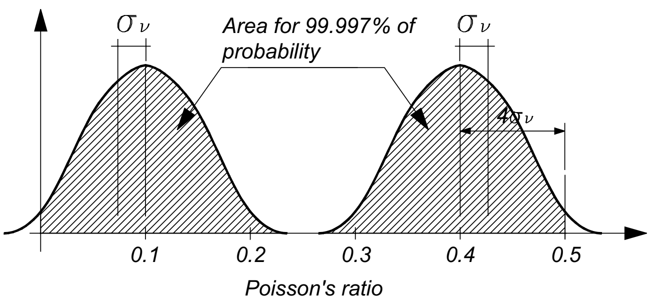

Fig. 16. Demonstration of application limit.

As it were, Fig. 15 and Eq. (43) suggest that the stochastic field must be in the state of a point (v_o, α) under a specific limit line, determined depending on the peak factor K_p, for the proposed method (or possibly all the other analysis schemes when considering the uncertain Poisson's ratio which is assumed to follow Gaussian distribution) to give theoretically reasonable results. Furthermore it has to be noted that the application limit of this sort is inevitable in all the other analysis schemes, if any, because of the physical constraint in the Poisson's ratio, whether it is assumed as Gaussian or non-Gaussian. In addition, since the probabilistic distributions for the mean Poisson's ratio near 0.5 or 0.0 are expected to be severely different from that with mean Poisson's ratio in the intermediate range, special concerns have to be taken for these cases.

6. Conclusions

In this paper, to deal with the effect of randomness in Poisson's ratio on the response variability in plane structures, a new formulation in the context of weighted integral stochastic finite element method is proposed for the plane strain and plane stress states. With close investigations on the constitutive relations, the constitutive matrix is decomposed into several sub-matrices by way of employing the polynomial expansion on the elements of original constitutive matrix. To derive the final formulae for statistical results, the transformation of expectations on the power stochastic field function into new auto-correlation functions is established.

To verify the accuracy and efficacy of the proposed formulation, in-plane plate and beams are taken as examples. For plate type in-plane structure, the maximum response variability is revealed to be over the COV of stochastic field σ_{ff}, 0.1 in the example analysis, in case of plane strain and to be the same as σ_{ff} when in the state of plane stress. For beam structures, or structures with bending and shear behaviors, the response variability is evaluated to be relatively small, around 10% of COV of stochastic field for plane strain state and even smaller for plane stress state. In summary, the effect of Poisson's ratio on the response variability appears to be smaller in plane stress than in plane strain and when the structure includes bending and shear behaviors than when it does not. Contrary to the case of uncertain elastic modulus, the response variability is affected by the value of Poisson's ratio itself. In all the analyses, the proposed method shows reasonable agreement with classical MCS.

With proposed formulation, it becomes possible for the weighted integral stochastic finite element method to take into account of all the material constants (i.e., elastic modulus and Poisson's ratio) in its application, and it is expected that the evaluation of response variability due to multiple uncertain parameters, including not only the material ones but also the geometrical ones, will be possible in the near future.

Acknowledgements

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Appendix A

The binomial theorem states that

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \tag{A.1}$$

Therefore, each element of matrices $\mathbf{D}_{\mathrm{pss}}$ and $\mathbf{D}_{\mathrm{psn}}$ in Eqs. (1) or (2) can be expanded as follows:

A.1. For plane stress state

$$\frac{1}{1-v^2} = \frac{1}{1-v_o^2(1+f)^2} = 1 + v_o^2(1+f)^2 + v_o^4(1+f)^4 + v_o^6(1+f)^6 + \cdots = 1 + \sum_{l=0}^{\infty}\sum_{k=1,(2k\geqslant l)}^{\infty}\binom{2k}{l} v_o^{2k} f^l \tag{A.2}$$

$$\frac{v}{1-v^2} = \frac{v_o(1+f)}{1-v_o^2(1+f)^2} = v_o(1+f) + v_o^3(1+f)^3 + v_o^5(1+f)^5 + v_o^7(1+f)^7 + \cdots = \sum_{l=0}^{\infty}\sum_{k=1,(2k-1\geqslant l)}^{\infty}\binom{2k-1}{l} v_o^{2k-1} f^l \tag{A.3}$$

$$\frac{1}{1+v} = \frac{1}{1+v_o(1+f)} = 1 - v_o(1+f) + v_o^2(1+f)^2 - v_o^3(1+f)^3 + v_o^4(1+f)^4 - \cdots = 1 + \sum_{l=0}^{\infty}\sum_{k=1,k\geqslant l}^{\infty}(-1)^k\binom{k}{l} v_o^k f^l \tag{A.4}$$

A.2. For plane strain state

$$\frac{v}{1+v} = v_o(1+f) - v_o^2(1+f)^2 + v_o^3(1+f)^3 - v_o^4(1+f)^4 + - \cdots = \sum_{l=0}^{\infty}\sum_{k=1,k\geqslant l}^{\infty}(-1)^{k+1}\binom{k}{l} v_o^k f^l \tag{A.5}$$

$$\frac{1-v}{1+v} = \frac{1}{1+v} - \frac{v}{1+v} = 1 - 2v_o(1+f) + 2v_o^2(1+f)^2 - 2v_o^3(1+f)^3 + 2v_o^4(1+f)^4 - + \cdots = 1 + 2\sum_{l=0}^{\infty}\sum_{k=1,k\geqslant l}^{\infty}(-1)^k\binom{k}{l} v_o^k f^l \tag{A.6}$$

$$\frac{1-2v}{1+v} = \frac{1}{1+v} - \frac{2v}{1+v} = 1 - 3v_o(1+f) + 3v_o^2(1+f)^2 - 3v_o^3(1+f)^3 + 3v_o^4(1+f)^4 - + \cdots = 1 + 3\sum_{l=0}^{\infty}\sum_{k=1,k\geqslant l}^{\infty}(-1)^k\binom{k}{l} v_o^k f^l \tag{A.7}$$

$$\frac{1-v}{1-2v} = \frac{1}{1-2v} - \frac{v}{1-2v} = 1 + v_o(1+f) + 2v_o^2(1+f)^2 + 4v_o^3(1+f)^3 + 8v_o^4(1+f)^4 + \cdots = 1 + \sum_{l=0}^{\infty}\sum_{k=1,k\geqslant l}^{\infty}2^{k-1}\binom{k}{l} v_o^k f^l \tag{A.8}$$

$$\frac{v}{1-2v} = \frac{1-v}{1-2v} - 1 = v_o(1+f) + 2v_o^2(1+f)^2 + 4v_o^3(1+f)^3 + 8v_o^4(1+f)^4 + \cdots = \sum_{l=0}^{\infty}\sum_{k=1,k\geqslant l}^{\infty}2^{k-1}\binom{k}{l} v_o^k f^l \tag{A.9}$$

where, $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.

Appendix B

To prove the convergence of equations (A.2)–(A.9), the Weierstrass M test (Arfken, 2000) is employed. If we can construct a series of numbers $\sum_{i=1}^{\infty} M_i$, in which $M_i|u_i(\mathbf{x})|$ for all $\mathbf{x}$ in any interval, and $\sum_{i=1}^{\infty} M_i$ is convergent, the series $\sum_{i=1}^{\infty} u_i(\mathbf{x})$ will be uniformly convergent in the interval under consideration.

To show the convergence of the series of functions of constitutive matrix, the first term for plane stress state, $1/(1-v^2(\mathbf{x}))$, is taken as an example. In this case, the series $\sum_{i=1}^{\infty} u_i(\mathbf{x})$ becomes

$$\sum_{i=1}^{\infty} u_i(\mathbf{x}) = 1 + \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} \binom{2k}{l} v_o^{2k} f^l(\mathbf{x}) = \sum_{i=1}^{\infty} \alpha_{i-1} f^{i-1}(\mathbf{x}) \tag{B.1}$$

After some manipulations, the general formula for coefficients α_k in (B.1) can be derived as

$$\alpha_k = \frac{v_o^{2k}}{(1-v_o^2)^{k+1}} A_k + \frac{1}{(1-v_o^2)^k} B_k; \quad (k = 0, 1, 2, \ldots) = \alpha_k^a + \alpha_k^b \tag{B.2}$$

where

$$\begin{aligned} A_k &= \sum_{i=1}^{k+1} m_{ki} \binom{2k-2(i-1)}{k}; \quad k \geqslant 2(i-1) \\ B_k &= \sum_{l=0}^{L} \sum_{i=1}^{k+1} m_{ki} \binom{k_s+2l-2(i-1)}{k} v_o^{k_s+2l}; \quad k_s + 2l - 2(i-1) \geqslant k \end{aligned} \tag{B.3}$$

In B_k, $L = L_q - 1$, where $k = 2L_q + n (n = 0 \text{ or } 1)$, and $B_k = 0$ if $L < 0$. The k_s denotes the first power number determined as k (if k is an even number), and $k+1$ (if k is an odd number). The constants m_{ki}s are derived as in Table 1. The coefficients α_k s are evaluated as follows:

$$\begin{aligned} &\alpha_0 = \frac{1}{1-v_o^2}, \quad \alpha_1 = \frac{2v_o^2}{(1-v_o^2)^2}, \quad \alpha_2 = \frac{4v_o^4}{(1-v_o^2)^3} + \frac{v_o^2}{(1-v_o^2)^2} \\ &\alpha_3 = \frac{8v_o^6}{(1-v_o^2)^4} + \frac{4v_o^4}{(1-v_o^2)^3}, \quad \alpha_4 = \frac{16v_o^8}{(1-v_o^2)^5} + \frac{v_o^4 + 11v_o^6}{(1-v_o^2)^4} \\ &\qquad \vdots \end{aligned} \tag{B.4}$$

Table 1
Constants m_{ki}

α_k	2^k	m_{ki} $(i = 1, 2, \ldots, k+1)$
α_0	1	1
α_1	2	1 −1
α_2	4	1 −2 1
α_3	8	1 −3 3 −1
α_4	16	1 −4 6 −4 1
α_5	32	1 −5 10 −10 5 −1
α_6	64	1 −6 15 −20 15 −6 1
α_7	128	1 −7 21 −35 35 −21 7 −1
⋮	⋮	⋮
		$\sum_{i=1}^{k+1} \lvert m_{ki} \rvert = 2^k$

Noting that if $s_k^{(1)} > s_k^{(2)}$ and series $\{s_k^{(1)}\}$ is convergent, then series $\left\{s_k^{(2)}\right\}$ is also convergent, and that the coefficients α_is in (B.2) and (B.4) are consists of two parts, we can construct a set of constant a_i and b_i as

$$\begin{aligned} a_k &= \frac{2^k v_o^{2k}}{(1-v_o^2)^{k+1}} (= \alpha_k^a); \quad k = 0, 1, 2, \ldots \\ b_{2k} &= \frac{2^k v_o^{2k}}{(1-v_o^2)^{2k}} (> \alpha_{2k}^b); \quad k = 1, 2, 3, \ldots \\ b_{2k+1} &= \frac{2^{k+1} v_o^{2k+1}}{(1-v_o^2)^{2k+1}} (> \alpha_{2k+1}^b); \quad k = 1, 2, 3, \ldots \end{aligned} \tag{B.5}$$

where $b_0, b_1 = 0$ and α_k^a, α_k^b are defined in (B.2). Thus, M_is are established as follows:

$$M_{i+1} = a_i + b_i \geqslant |u_{i+1}(\mathbf{x})|; \quad i = 0, 1, 2, \ldots \tag{B.6}$$

where $f(\mathbf{x})$ is taken as 1.0 since $|f(\mathbf{x})| < 1.0$. Therefore only the convergence of $\sum_{i=1}^{\infty} M_i$ has to be proved. As seen in (B.5) and (B.6), $\sum_{i=1}^{\infty} M_i$ consists of three geometric series: $\{a_k\}$, $\{b_{2k}\}$ and $\{b_{2k+1}\}$. Since the sum of geometric series $\{s_i\}$ is given as $s_0/(1 - r_s)$, the converged value of $\sum_{i=1}^{\infty} M_i$ is evaluated as

$$\sum M_i = \frac{1}{1 - 3v_o^2} + \frac{8v_o^3 - 4v_o^5}{(1 - v_o^2)(1 - 4v_o^2 + v_o^4)} \tag{B.7}$$

Therefore the series $\sum_{i=1}^{\infty} u_i(\mathbf{x})$ is uniformly convergent. In an analogous way, the convergence of the other coefficients β_i, γ_i and κ_i, θ_i, ρ_i, δ_i, ε_i can be established with ease.

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